SPRING **C·ODE·E** 1993

Announcements

Invitation to Authors!

We (the editors and board of **C•ODE•E**) would like to invite you to send us your letters, responses to articles, comments on experiments, articles or software reviews. The mailing address and e-mail address are on the back cover of this issue.

Last Call for Workshops

In the summer of 1994 the Consortium will be hosting a final summer of workshops. Details will be published in future issues, but you should know that these “Last Chance” workshops will be taking place at RPI in Troy, New York, and at West Valley College in Saratoga, California.

1993 Workshops Successful

The 1993 summer workshops are over, and have been a real success. See future issues for articles about the workshops and some of the experiments developed.

Back Issues

Back issues are now available by anonymous FTP! Use ftp.hmc.edu in the directory /pub/CODEE. All issues are in PostScript, and are identified by season and year (for example, this issue would be called spring93.ps).

The Painlevé Transcendent

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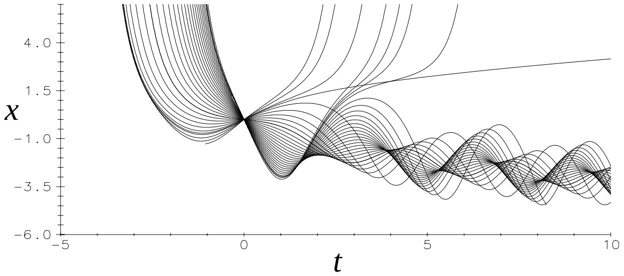


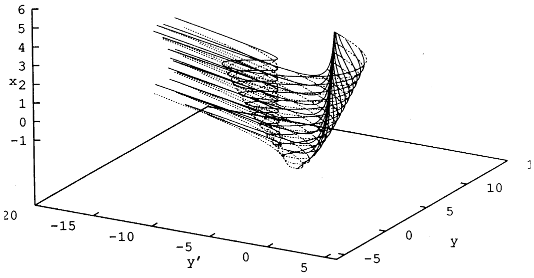
The Painlevé Transcendent: The differential equation $x'' = x^2 - t$ has interested mathematicians since early in the 20th century. It is a variation of the first Painlevé transcendent, one of a group of nonlinear second-order differential equations having solutions whose only movable singularities are poles. These equations cannot be solved in terms of elementary or classical functions like the trigonometric, Bessel, or Legendre functions[1]. For decreasing t, the solutions of the first transcendent all tend to infinity in finite time, and a great deal is known about the spacing of their singularities. In this context, a solution $x(t)$ has a singularity at $t = T$ if $|x(t)| \to \infty$ as $t \to T$.

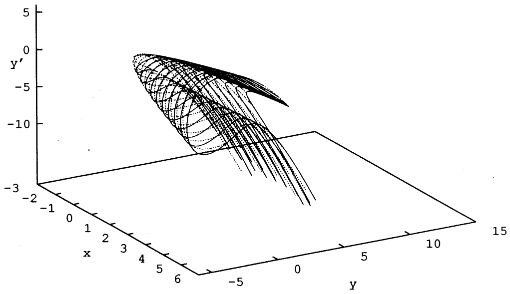
For increasing t, three distinct types of behavior are possible. Some solutions become infinite at finite values of t. If one looks at the 3-dimensional (t, x, x') space, however, one finds a solid region of solutions $x(t)$ that oscillate about the graph of $x = -\sqrt{t}$ and approach the graph asymptotically as $t \to \infty$. The figure on the previous page shows a bundle of the oscillatory solutions through the point $x(0) = 0$. The slopes of these solutions vary from $x'(0) \cong -4.2$ to $x'(0) \cong 1.4$. This picture can be easily reproduced by almost any ODE software by using the non-autonomous planar system $x' = y, y' = x^2 - t$ or the equivalent autonomous system $x' = y$, $y' = x^2 - t$, $t' = 1$.

With the initial conditions $t = 0, x = 0$, one can vary $y(0) \equiv x'(0)$ both positively and negatively until the solutions fly off to infinity in finite time. Graphing the bounding surface S of the region of oscillatory solutions is an altogether more difficult problem.

In a 1984 paper[2], Holmes and Spence considered a boundary value problem for this equation, which arose in a physical problem in fluid flow. The flow problem required a solution with $x(0) = 0$ and x asymptotic to $x = \sqrt{t}$ as $t \to \infty$. They proved the existence of a unique solution with $x'(0) > 0$, but they were unable to show that there is also a unique solution with $x'(0) < 0$. A recent paper by Hastings and Troy[3] proved that this is indeed the case.

Plotting the Bounding Surface: I have been able to show that the solutions asymptotic to $x = \sqrt{t}$ form a one-parameter family of curves, uniquely parameterized by their intersection with the parabola $x^2 = t$, at a point where the solution curve has positive slope. The family forms the surface of the solid region of oscillatory trajectories. This surface is very hard to characterize, and even harder to graph, since the trajectories which form it are highly unstable. The slightest numerical error produces either an oscillatory solution or one tending quickly to infinity. Producing an analytic description of this surface has been a priority for me ever since I watched John Hubbard experimenting with his **MacMath** differential equations software, trying to picture the shape of the region filled out by the oscillatory solutions. The formal results of this research will be ready for publication shortly, but two different views of the repelling surface are shown here to pique your curiosity.





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1 H.T. Davis, **Introduction to Nonlinear Differential and Integral Equations**, Dover (1962).

[2] Philip Holmes and David Spence, *On a Painlevé-type boundary value problem*, **Quarterly J. Mech. Appl. Math.**, *Vol. 37,* (1984), pp. 525-538.

[3] S.P. Hastings and W.C. Troy, *On some conjectures of Turcotte, Spence, Bau, and Holmes*, **SIAM J. Math. Anal.**, Vol. 20, (1989), pp. 634-642.