- *"How to Balance a Stick When Walking in a Straight Line"* by Boyd Cardon, Gene Enneking, Fred Wilke
- *"Interactions and Reactions: An Introduction to Analysis of 2-D Nonlinear Dynamics"* by Jeffery Palmer and Wei-Jen Harrison.

St. Olaf College

(June 20-25, 1993)

Arnold Ostebee and Steven Kennedy organized and hosted the workshop, Computer Experiments in Ordinary Differential Equations, one of the six workshops supported by the NSF and run by the Consortium for Ordinary Differential Equations during the summers of 1992, 1993, and 1994. The twenty-five participants used a variety of solvers on a variety of platforms, reported that they had a great time and learned a lot, and designed a number of interesting experiments for an ODE computer lab. Here are the titles of some of the experiments (and their authors):

- *"The Average Distance Between Points in a Disk"* by Steven R. Dunbar & Richard Bernatz
- *"Mining Carbonite in the Degan System"* by Jerold Mathews
- *"A Plant-Herbivore Model; or Why is the World Green?"* by Jon L. Johnson & Lisa Holden
- *"Can One Exceed Terminal Velocity?"* by Fink, Freeman, & Hampton.

Some of the experiments from these workshops will appear in later issues of **C•ODE•E**.

A Polluted Lake

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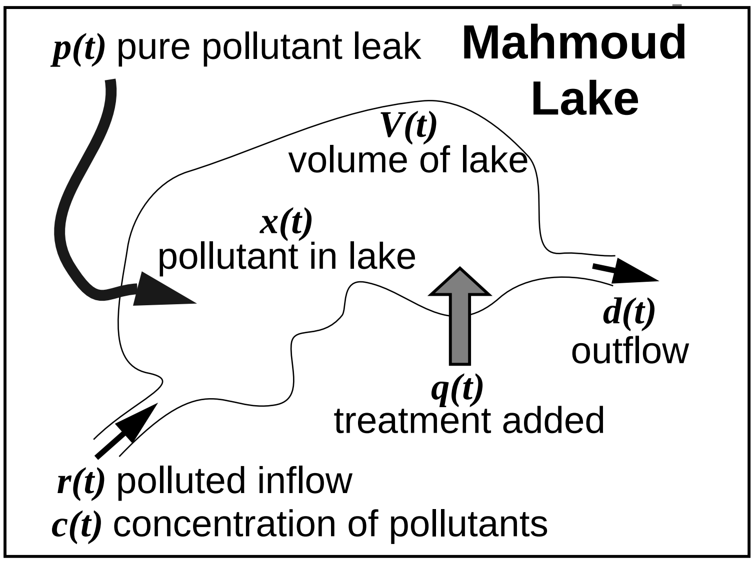
Computers and ODEs: When teaching an introductory differential equations course, we usually use suitably cooked functions and limit the applications to those that result in not-so-hard integrals. Thanks to recent computer technology and the mathematical software library (**MacMath, Mathematica, Maple, Derive,...**), we can consider more realistic problems, and let the computer do many of the long, tedious computations. It is still very important that the students know how, and why, the methods work. Then, they are motivated to use these computational packages to save time and effort, and concentrate instead on the ideas and the theory behind the computations.

Computers also make students excited and willing to learn more math. It is believed that introducing our students to easy-to-use software is best accomplished at the differential equations level. By then, the students already have had a good dose of mathematical analysis and logical thinking, which in turn gives them a chance to appreciate the role mathematics plays in our life.

In this paper, we will consider the familiar problem of pollution in a lake in a relatively general setting (see [1] for another version of this model). A computer package will be used to find and investigate the solution of the differential equation that models the problem with different initial conditions. By bringing this kind of technology into the



classroom, we will show that many hard topics and ideas can be discussed, even in an elementary differential equations course, and a wider range of students will be able to grasp the material better.

A Dirty Lake: The sketch above shows the lake with two sources of pollution, an input stream of clean-up chemicals, and a drainage stream. The variables to be used and their meanings are listed here:

- ***V(t)***: The volume of the lake in gallons.
- ***x(t)***: The amount of pollutant in the lake at time t, measured in parts. The pollutant is assumed to be evenly distributed throughout the lake. Therefore the concentration of pollutant in the lake at time t is $x(t)/V(t)$.
- ***p(t)***: The rate at which pure pollutant comes into the lake in parts/second (e.g., from an oil leak).
- ***r(t)***: The rate at which a mixture of pollution and water enters the lake. The concentration of pollution in the mixture is ***c(t)*** (upstream factories, small ships, etc.).
- ***q(t)***: A treatment chemical T added to the lake to reduce pollutants, expressed in gallons/second. One gallon of this treatment T dissolves α parts of the pollutant. The units of α are gallons^{-1}.
- ***d(t)***: The rate the lake's water drains, from the outlet in gallons/second.
- ***x'(t)***: The rate of change of the amount of pollution in the lake expressed in parts/second. This is then the rate at which pollution enters the lake minus the rate at which it is removed by drainage or treatment.

Hence we have the following first order linear rate equation

$$x'(t) = [r(t)c(t) + p(t)] - \left[\alpha q(t)x(t) + \frac{d(t)x(t)}{V(t)}\right]$$

which reduces to

$$x'(t) + E(t)x(t) = r(t)c(t) + p(t)$$

where

$$E(t) = \left[\frac{d(t)}{V(t)} + \alpha q(t)\right]$$

Solving this, and letting $\xi = \int E(t)\,dt$, we have a formula for the amount of pollutant:

$$x(t) = e^{-\xi}\int e^{\xi}[r(t)c(t) + p(t)]\,dt$$

Now What? This is the exact solution of the differential equation. The integrals might be hard or even impossible to be evaluated exactly. However, we can always use numerical techniques to find an approximation to a specific initial value problem. A short cut is to directly attack the differential equation, and use a finite difference method e.g., Euler, Taylor, Runge-Kutta, Adams-Bashforth, etc. (the choice of the method depends on the equation). **MacMath** (as well as **Mathematica** and **Maple,** to name a few) does that for us; **MacMath** also allows us to choose a specific numerical method.

It is, however, very important that we know what the programs are doing to find the approximations to the solution, and how far we are from the exact solution. Algorithms that are used in these computer packages, are usually studied in detail in a numerical analysis course, while the conditions for the existence and uniqueness of the solution, as well as its local or asymptotic behavior, are investigated in an advanced ODE course. Here, we are mainly interested in studying the solutions of some differential equations that would be impossible to look at without the help of these computer packages.

Solver to the Rescue: MacMath (see [2]) is a software program that is developed mainly for the purpose of solving DEs, and it has many user friendly features. It is capable of handling many differential equations that model a wide variety of realistic applications, and is an excellent choice for use in a computing laboratory for the introductory ODE course. *(Editor's Note: The author included several very nice graphs generated by MacMath. For technical reasons, a local solver was instead used to generate the graphics for C•ODE•E).*

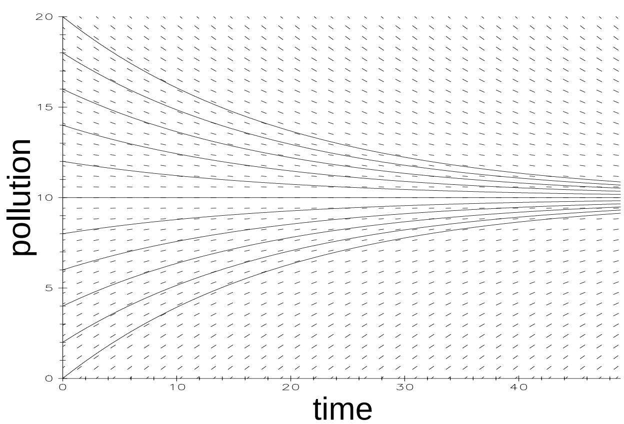
Before we consider some examples of the pollution problem, it is worth mentioning that in the above model we did not consider rainfall or evaporation. Considering these two factors leads to a system of three differential equations *(Editor's Note: see the next article in this issue of* C•ODE•E*)*.

Examples: In all the following examples, $V(0)$ will always be 100 gallons (a mud puddle, not really a lake!), with initial pollution $x(0)$ gallons. If gallons seems to you to be the wrong units for this model, interpret everything in terms of, say, cubic decameters, as in Noonburg's article.

Example 1:

$$r(t) = d(t) = 5 \qquad c(t) = 0.1$$
$$p(t) = 0 \qquad q(t) = 0$$

In this example, we have no pure pollution, no treatment, and 10% polluted water flow into the lake; fill and drain rates are equal, so $V(t)$ is constant. From these solu-



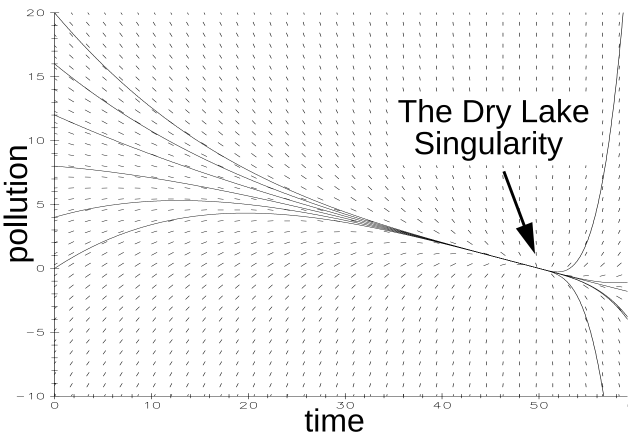
tion curves, it is quite easy to investigate whether the pollution increases or decreases, the steady-state, the dependence on the initial level of pollution of the water, and so on. In this case, since $r(t)$, $d(t)$, $c(t)$, $p(t)$, and $q(t)$ are all constant, the analytical solution of the differential equation is easily found to be

$$x(t) = 10 + (x(0) - 10)\, e^{-t/20} .$$

Example 2: Change $d(t)$ to 7 in Example 1 hence,

$$V(t) = 100 + 5t - 7t = 100 - 2t \quad .$$

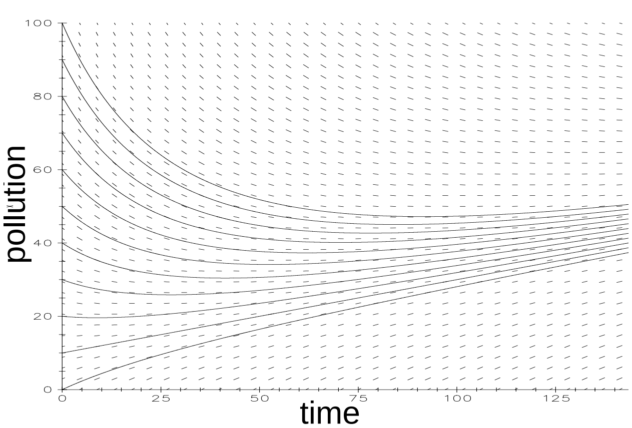
In this example, we have no pure pollution, no treatment, and 10% polluted water flows into the lake, at a rate of 5 gal/sec, and the lake drains at a rate of 7 gal/sec.



Again, from the above solution curves, we can answer the same kind of questions as in Example 1. We notice that at a certain time, the lake will be empty since $d(t) > r(t)$. After this time the differential equation does not model the problem any more.

Example 3: Change *d(t)* to 3 in Example 1; hence,

$$V(t) = 100 + 5t - 3t = 100 + 2t \quad .$$



In this example, we have no pure pollution, no treatment, and 10% polluted water flows into the lake, at a rate of 5 gal/sec, and the lake drains at a rate of 3 gal/sec.

Again, from the above solution curves, we can answer the same kind of questions as in Example 1. The analytical solution of the differential equation for this example is found to be

$$x(t) = \frac{1}{10}(100 + 2t) + \frac{(x(0) - 10)\,100^{3/2}}{(100 + 2t)^{3/2}}$$

Example 4:

$$r(t) = \sqrt{t} \qquad d(t) = t$$
$$c(t) = \sin(t) \qquad p(t) = 0.05t$$
$$q(t) = 0 \qquad \alpha = 0.05$$

Here, the parameters of the problem are functions of time. We have pure pollution as well as polluted water that flows into the lake with a variable concentration of pollutants.When *c(t)* is negative, that means it represents treatment of the pollution already in the lake. *(No graph for this one! See the next graph, it is very similar.)*

The analytical soultion to this problem is therefore:

$$V(t) = 100 + \int_0^t \sqrt{s}\,ds - \int_0^t s\,ds + \int_0^t 0.05s\,ds$$

$$V(t) = 100 + \frac{2}{3}t^{3/2} - \frac{1}{2}t^2 + \frac{0.05}{2}t^2$$

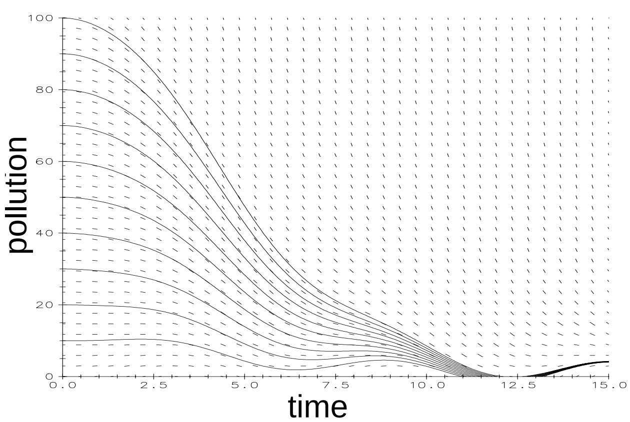
Example 5:

$$r(t) = \sqrt{t} \qquad d(t) = t$$
$$c(t) = \sin(t) \qquad p(t) = 0.05t$$
$$q(t) = t \qquad \alpha = 0.05$$

Finally, we introduce the "clean-up" chemical. In this case, we see that

$$V(t) = 100 + \frac{2}{3}t^{3/2} + 0.025t^2 \quad .$$

Also in this example, all the parameters of the problem are functions of time. Each gallon of the treatment dissolves .05 gallons of the pollutant.*)*



Cleaner Than Clean: The models of the last two examples must be treated with care since the "concentration" $c(t)$ is allowed to be negative (thus, modeling "cleanup" of pollution already in the lake). Students should observe that once the pollution level $x(t)$ reaches 0 then the models must be adjusted (how?) so that $x(t)$ doesn't become negative. In particular, the lake of Example 5 reaches a state of cleanliness that even the EPA would find incredible!

(Editor's Note: Compare the results of this model to those in the next article!)

1. W.E. Boyce and R.C. DiPrima. **Elementary Differential Equations and Boundary Value Problems**. 5th ed. John Wiley, 1992

2. J.H. Hubbard and B.H. West, **MacMath: A Dynamical Systems Software Package for Macintosh**. Springer-Verlag, 1992 ❑

More on the Polluted Lake

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(Editor's Note: Noonburg adds rainfall and evaporation to the model explored by fath El-Den earlier in this issue.)

Setting the Scene: A small lake, Lake Noonburg, is fed by 3 polluted streams and has a single outlet stream. The surface area of the lake is assumed constant at 4×10^5 square meters. To keep the units easy to work with, length will be measured in dekameters (Dm), with 1 Dm = 10 meters. Thus the surface area is 100Dm x 40Dm = 4000 sq.Dm. Assuming an average depth of 1/4 Dm (~ 7.5 ft.), the initial volume of water in the lake is 1000 cu.Dm

The rate of flow in stream i is r_i (cu.Dm/day) and concentration of pollutant in input stream i is p_i (in "parts" per cu.Dm).

Rate Equations: In a compartmental model, it is assumed that the change in the amount of any substance in the compartment is equal to the volumetric rate at which it flows in minus the rate it flows out. Therefore, if $x(t)$ is the amount of a certain pollutant in Lake Noonburg at time t, then we have that $x(t)$ = rate pollution flows in - rate pollution flows out, or

$$x'(t) = (r_1 p_1 + r_2 p_2 + r_3 p_3) - \frac{r_4 x(t)}{V(t)}$$

$$= R - \frac{Sx(t)}{V(t)}$$