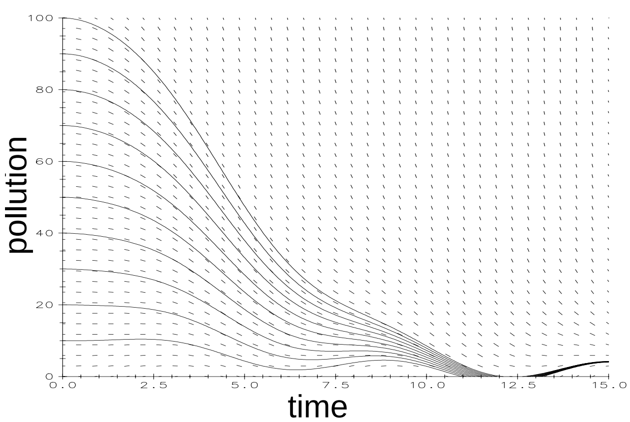
Finally, we introduce the "clean-up" chemical. In this case, we see that

$$V(t) = 100 + \frac{2}{3}t^{3/2} + 0.025t^2 \quad .$$

Also in this example, all the parameters of the problem are functions of time. Each gallon of the treatment dissolves .05 gallons of the pollutant.*)*



Cleaner Than Clean: The models of the last two examples must be treated with care since the "concentration" $c(t)$ is allowed to be negative (thus, modeling "cleanup" of pollution already in the lake). Students should observe that once the pollution level $x(t)$ reaches 0 then the models must be adjusted (how?) so that $x(t)$ doesn't become negative. In particular, the lake of Example 5 reaches a state of cleanliness that even the EPA would find incredible!

(Editor's Note: Compare the results of this model to those in the next article!)

1. W.E. Boyce and R.C. DiPrima. **Elementary Differential Equations and Boundary Value Problems**. 5th ed. John Wiley, 1992

2. J.H. Hubbard and B.H. West, **MacMath: A Dynamical Systems Software Package for Macintosh**. Springer-Verlag, 1992 ❏

More on the Polluted Lake

V. Anne Noonburg

University of Hartford
West Hartford, CT 06117
noonburg@uhavax.hartford.edu

(Editor's Note: Noonburg adds rainfall and evaporation to the model explored by fath El-Den earlier in this issue.)

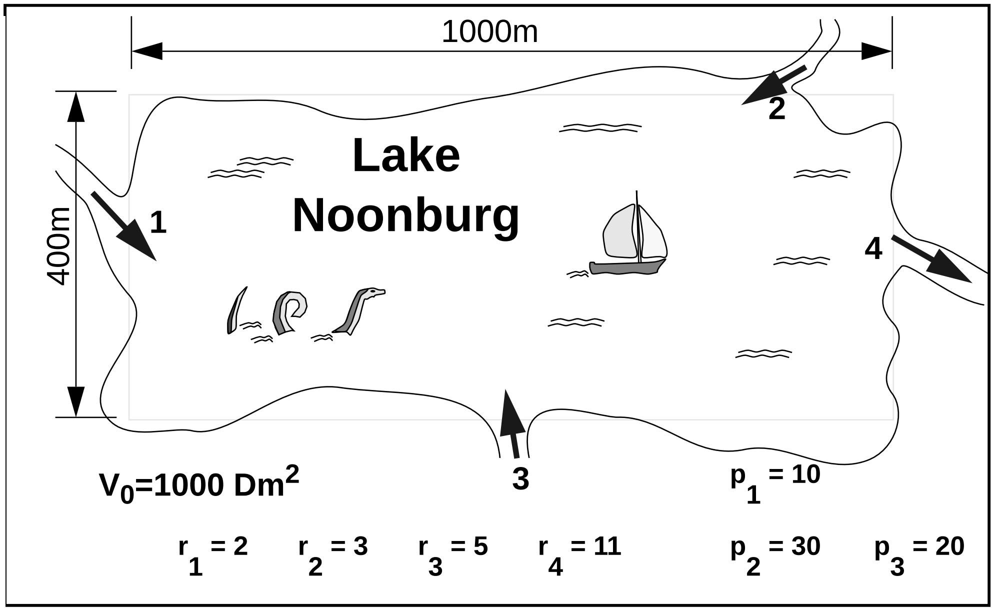
Setting the Scene: A small lake, Lake Noonburg, is fed by 3 polluted streams and has a single outlet stream. The surface area of the lake is assumed constant at $4x10^5$ square meters. To keep the units easy to work with, length will be measured in dekameters (Dm), with 1 Dm = 10 meters. Thus the surface area is 100Dm x 40Dm = 4000 sq.Dm. Assuming an average depth of 1/4 Dm (~ 7.5 ft.), the initial volume of water in the lake is 1000 cu.Dm

The rate of flow in stream i is r_i (cu.Dm/day) and concentration of pollutant in input stream i is p_i (in "parts" per cu.Dm).

Rate Equations: In a compartmental model, it is assumed that the change in the amount of any substance in the compartment is equal to the volumetric rate at which it flows in minus the rate it flows out. Therefore, if $x(t)$ is the amount of a certain pollutant in Lake Noonburg at time t, then we have that $x(t)$ = rate pollution flows in - rate pollution flows out, or

$$x'(t) = (r_1p_1 + r_2p_2 + r_3p_3) - \frac{r_4x(t)}{V(t)}$$

$$= R - \frac{Sx(t)}{V(t)}$$



where $V(t)$ is the volume of the lake at time t, R = 210, and S=11. The quantity $x(t)/V(t)$ represents the concentration of pollutant, $C(t)$, in the lake (expressed in "parts" per cu.Dm).

To obtain the volume $V(t)$ at time t, the following similar equation can be constructed: dV/dt = rate water flows in - rate water flows out = $(r_1 + r_2 + r_3 - r_4)$ + (total rainfall - total evaporation).

Rainfall and Evaporation: The average daily rainfall and average daily evaporation rates (also measured in cu.Dm per square Dm of surface area per day) are given below. They are assumed to be periodic functions with periods 365/2 and 365 days, respectively.

The average daily rainfall and evaporation rates (measured in cu. Dm per sq. Dm of surface area per day) are sinusoids of respective amplitudes R_1 and R_2 about their respective averages C_1 and C_2. Their periods, in order, are 182.5 days and 365 days, with phase shifts of 75 and 135 days to reflect spring/fall rainy seasons and a high summer evaporation rate.

Daily rainfall (Dm/day):

$$C_1 + R_1 \sin(2\omega(t-75)),\ \omega = 2\frac{\pi}{365}$$

Total rainfall into the lake (cu. Dm/day):

$$4000(C_1 + R_1 \sin(2\omega(t-75))) = P + A\sin(2\omega(t-75))$$

Daily evaporation (Dm/day):

$$C_2 + R_2 \sin(\omega(t-135))$$

Finally, the total evaporation from the lake:

$$4000(C_2 + R_2 \sin(\omega(t-135))) = Q + B\sin(\omega(t-135))$$

Concentration of Pollutants: Since the volume varies with time, one is really more interested in the concentration of pollutant in the lake than in the total amount of pollutant. To write a differential equation for the concentration, let $C(t) = x(t)/V(t)$. Then

$$C'(t) = \frac{V(t)x'(t) - x(t)V'(t)}{V^2(t)}$$

$$= \frac{V(t)}{V^2(t)}\left[R - \frac{Sx(t)}{V(t)}\right] - \frac{x(t)}{V(t)}\frac{V'(t)}{V(t)}$$

$$= \frac{R - SC(t) - C(t)V'(t)}{V(t)}$$

This leads to the following system:

$$V' = F + A\sin(2\omega(t-75)) - B\sin(\omega(t-135))$$

$$C' = \{R - C[S + F + A\sin(2\omega(t-75)) - B\sin(\omega(t-135))]\}$$

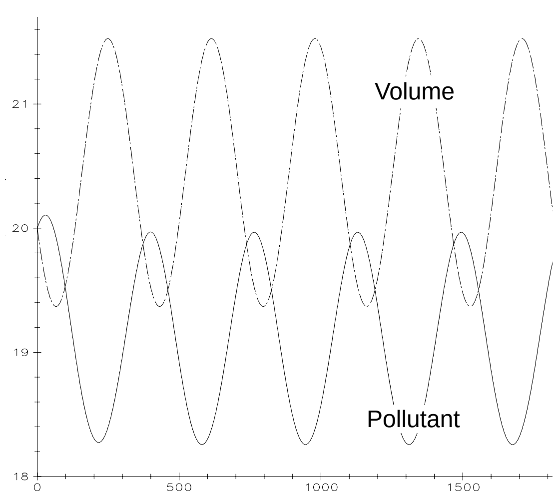
$$F = r_1 + r_2 + r_3 + r_4 + P - Q$$

Let $C_1 = 3 \times 10^{-4}$ Dm/day, $C_2 = 5 \times 10^{-5}$ $R_1 = 2.5 \times 10^{-4}$, and $R_2 = 4 \times 10^{-5}$, and we have that $F = 0$!

Sample Problems: The following problems should be done using an ODE solver. Attach graphics.

- 1. Set up and solve this system with the given parameter values. Let t go from 0 to 1825 days ($\cong 5$ years), with $\Delta t = 2$ days. Assume an initial concentration of 20 parts/cu.Dm for the pollutant in the lake.
- 2. Describe how the concentration of the pollutant varies over time. Does it reach some kind of steady state? Explain.

(Editor's Note: I used a local software package to solve this problem. Here you can see the regular rise and fall of pollution levels in the lake and the simultaneous change in lake volume.)



- 3. How would the equations change if the rainwater contains a pollutant concentration of p_r parts/cu.Dm Let p_r= 5, and redo #1 and #2.
- 4. Which is more damaging, a doubling of the pollutant level in stream 2 or in stream 3? (Try both cases, p_r = 0 and p_r = 5). Provide graphs and an explanation.

5. Make up a totally new hypothesis of your own, and test it. Provide graphs to back up your conclusions, and state carefully any changes you make in the parameters. ❑

Discovering Differential Equations

Margie Hale

Stetson University

DeLand, FL 32720

mhale@stetson.bitnet

While teaching at Ithaca College I had the opportunity to develop and teach a differential equations course in a laboratory equipped with ten Sparc workstations. Each class contained about ten students majoring in various liberal arts disciplines. The approach I took was to replace some calculation with modeling and qualitative analysis, and my lecturing with student discovery.

The principal software available was **MATHEMATICA**, chosen by the department to enhance many of the upper level mathematics courses. I wrote some programs specifically for a differential equations course, and I supplemented with BASIC programs for certain applications. No knowledge of computers was assumed at the start of the course.

Syllabus: Allowing students to experiment and to discover is a lengthy process.