ter were valuable. Both are related to keeping intellectual control in the classroom. First, I hired a computer-savvy student assistant to be in the laboratory during every class period. The assistant can respond to the student who pushes the wrong button while you're talking, leaving you free to direct the rest of the class. The assistant can call for help when the system crashes while you continue to focus the class on the subject matter. Second, each class period choose a student to sit at the machine connected to the projector instead of your sitting there. This way you can have demonstrations and still not interrupt the flow of your message.

Structuring The Laboratory: Here is a final word on the laboratory environment. The laboratory at Ithaca College is roomy, with each computer sitting on a desk which allows room for books and notebooks too. The instructor can get to every station for a personal discussion with the one or two students there, and to see their work. The monitors are lowered a few inches so that the students can easily see the professor and each other. These are all good points which contribute to a positive learning environment. The major flaw of that laboratory (as of 1992) was its lack of air conditioning. Computers like cold, and (being ornery) generate heat, and air conditioning is expensive.

Why MATHEMATICA? MATHEMATICA was chosen, after much deliberation, for its power and its adaptability to a variety of courses. It is superb at drawing single graphs, slower at drawing multiple graphs in color. As far as I know, it cannot draw graphs in "real time": you see only the finished product. This is a definite disadvantage in drawing phase-planes; one would like to see the dynamics in a dynamical system. The BASIC programs I wrote mostly addressed this problem. (A few were written so the students would have access to the code.) Probably one of the other software systems is better for use in a differential equations class only.

MATHEMATICA syntax is not friendly for the beginning computer student, and I had to provide a three-page reference sheet to help students with the commands they would need for the course.

Afterword: I am currently interested in collecting samples of specific questions that instructors ask their students in a computer environment. These should be questions which help a student make the connection between computer output and abstract idea. If you wish to share such questions, from any subject matter, please send them to me at the address above.

(Editor's Note: Hale's request is one that touches all of us teaching ODEs. Send **C•ODE•E** *a copy of your suggestions for possible publication.)* ❑

Differential Equations, Football, and Chase Problems

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I first learned of this problem in an article in the CASE Newsletter by Elgin Johnson and Jerold Mathews [1], describing the ``Project Based Calculus" at Iowa State University. It's also a classic problem which can be found in the text by Simmons [2].

College football is a favorite American

spectator sport, so a differential equations project involving football is almost guaranteed to generate student interest. It's especially popular here at the University of Nebraska-Lincoln, where Husker football is a tradition. The following is a classic chase problem in differential equations posed as ``final seconds cliff-hanger":

Go State! It was the day of the big game, NU against Enormous State University, on national TV, the fictional collegiate championship, the works! The Huskers led 24-21 late in the 4th quarter. The ESU Bruisers had the ball on their own 25. Bart Simpson, star running back for the Bruisers, burst through the Husker line. Looking up as he crossed the 25, he saw 75 yards of open field between him and the goal line. "Here comes six points, the game, the title, a first round draft choice, and a lucrative career in the NFL!" thought Bart.

Thirty yards to Bart's left, also on the 25 yard line, Husker linebacker Matt E. Mattics saw Bart break through. Due to excellent scouting reports, Matt knew he was faster than Bart. In fact, Bart could run 7.5 yards/second while Matt could do 8.4 yards/second. Hence, given enough time, Matt knew he could run Bart down. Matt immediately set out in pursuit of Bart. He figured the best way to catch his opponent and to save the game was to adjust his path constantly so that he was always running right at Bart; that is, run in such a path that at each instant his velocity vector was pointing directly at Bart.

As Matt started out on his pursuit, a groan immediately rose from the special block of seats reserved for the 221H Differential Equations class. Why did the students groan?

Matt Makes the Model: For convenience and greater generality, label Matt's starting point as (0,0), with the x-axis along the common starting yard-line. Matt's position at any time is $(x(t), y(t))$. Bart's initial position will be $(a,0)$. We will assume that Bart runs straight for the goal line which is b yards down-field, so his path will be along the line $x = a$. We will set v_q (the q is for quarry) as Bart's speed and v_p (the p is for pursuer) for Matt's speed.

The information about the direction of motion translates directly to

$$\frac{dy}{dx} = \frac{y(t) - v_q t}{x(t) - a}$$

The initial conditions are $y(0) = 0$ and $y'(0) = 0$.

Rearrange this, differentiate with respect to x, and then eliminate dt/dx by using Matt's velocity relationship in differential form

$$dx^2 + dy^2 = v_p^2 dt^2 \ .$$

The final equation is

$$(a-x)y'' = v\sqrt{1+(y')^2}$$

where $v = v_q/v_p$. By making the substitution $p = y'$, the equation reduces to a first-order separable equation. The solution for p can then be integrated again to yield y.

$$y = \frac{(a-x)^{v+1}a^{-v}}{2(v+1)} - \frac{(a-x)^{1-v}a^{v}}{2(1-v)} + \frac{av}{1-v^2}$$

Good Model, Bad Strategy: There are several good reasons for considering this problem in a class on differential equations.

- It's interesting, immediately understandable to students, and within their experience -- with an element of whimsy that students appreciate.
- This is a different modeling situation from many in differential equations as it uses very little physics. Therefore it may be well suited to biology students,

economics and business management students, and of course the traditional engineering and physical sciences student. This problem can also be used for an introduction to differential games.

- The equation is a nonlinear second-order equation which is nevertheless solvable in closed form by means of reduction to first-order equations. Thus the equation provides an alternative example in addition to the standard linear second-order equations usually presented in courses in differential equations. This also provides a bridge between first-order and second-order equations, and another example of solving first-order equations.
- The equation provides a simple, natural example of an equation with a singularity at $x = a$. The singularity is not serious as can be seen from the solution, but it may nevertheless be a challenge for some numerical solvers. This gives an opportunity for discussing the limitations of numerical solvers, or comparing the results of different solvers.
- The modeling provides an excellent platform for "what-if" scenarios and more modeling since one can easily vary the parameters, the initial conditions, and even the chase strategies. See the figure on the next page.
- The equation gives an example of the importance of non-dimensionalizing parameters, as it reduces the need for four parameters to two, the "shape parameter" $c = a/b$ and the speed ratio $v = v_q / v_p$. The two parameter groups are easy to understand, too, since the ratio of distances characterizes the relative shape of the situation and the velocity ratio is the parameter of main interest. Students should be asked to recast the problem in terms of v and c.
- The problem can be worked "by hand", although for some variations a computer algebra system may be handy. Numerical solution may also help understanding.
- Finally, the modeling gives a thorough review of calculus, arc-length, parametric equations, differentials, and vectors. Depending on how the problem is worked, the hyperbolic functions $\sinh x$ and $\cosh x$ may be introduced.

After working the problem, my students recognized that Matt's strategy was nearly the worst he could have reasonably chosen. Most students recognize that Matt needs to 'lead' his opponent in order to save the game with the last-second tackle.

What Should Matt Have Done? As one variation of strategy, imagine that the pursuer, Matt, always aims so that his velocity vector points at an interpolating point midway between the quarry, Bart, and the goal-line at $y = b$. Say the point aimed at is $\varepsilon b + (1 - \varepsilon) v_q t$. This introduces a strategy variable into the equation. If $\varepsilon = 0$, the situation is as before, while if $\varepsilon = 1$, the pursuer runs along the hypotenuse in order to make a last second goal-line tackle. If $c = a/b$ is the "shape" dimensionless parameter, then for $\varepsilon = 1$ it is easy to check that the required ratio of speeds for the pursuer to exactly capture the quarry is .

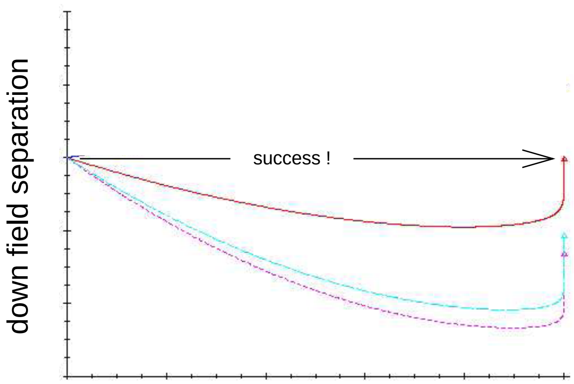
$$v = c / (\sqrt{1 + c^2}) \ .$$

It is clear that if the speed ratio v is less (so the pursuer is faster yet), then a strategy ε with $0 \le \varepsilon < 1$ may be employed. This could be used for instance, by the defensive coach to know how fast a player he needs to recruit for the team! A plot of the required speed ratio v as a function of the shape c and the strategy ε can be derived, but the details are messy.

An alternative strategy is to adopt a fixed amount d for the pursuer to lead the quarry. We can derive the initial value problem to be

$$\frac{dy}{dx} = \frac{y(t) - v_q t - d}{x(t) - a}$$

with $y(0) = 0$ and $y'(0) = d/a$. Once again, it is possible to derive the minimum required speed ratio v as a function of the shape c and the lead ratio $l = d/a$. The details are messy but can be performed with a computer algebra system such as Maple. Again, a transcript of the calculations is available from the author upon request.



The figure shows what would happen if Husker Matt chooses one of the three strategies. The first strategy is to run directly toward Bart. After ten seconds Bart has reached the goal and Matt trails him by several yards, clearly not a winning strategy for Matt (much less the rest of the team). The second strategy has Matt "leading" Bart by ten yards, but that, too fails as Matt will once more be several yards behind when Bart scores. The third option is a winner, though! This one uses the strategy parameter ε with $\varepsilon \approx 0.33$. This results in a goal-line tackle that puts the game in the bag for the Huskies!

My students have enjoyed this project very much, although it is challenging for most of them to derive the equation and analyze the solution. Many of the students say that this was their favorite project over the course of the semester. It's interesting for the instructor too, because there are a number of aspects to the project, each of which can be tailored and highlighted to suit the course and the syllabus.

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1 Elgin Johnston, Jerold Mathews,*"Project Based Calculus at Iowa State"* **Computer Algebra Systems in Education NEWSLETTER**, Number 11, September 1991.

2 George F. Simmons, **Differential Equations with Applications and Historical Notes, Second Edition**, McGraw-Hill, New York, 1991. ❑

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