Euler's method and the improved Euler's method, because that's what I'm trying to teach in an introductory course -- the ideas behind the basic methods, and the potential pitfalls of numerical methods in general. It would be great to have the option of using a sophisticated solver in **PHASER**, but having the basic methods is, in my opinion, much better. Besides, you can't fit everything on a single IBM PC diskette.

The Bestiary: No review of **PHASER** would be complete without mentioning the "manual". In addition to 15 well-designed tutorial lessons and a reference guide to the menus which contains just the right amount of detail, the book contains a short review of differential and difference equations and a short section on numerical methods in general. These are meant to supplement, not replace, a standard text on ODEs. Additionally, all of the stored equations that come with **PHASER** appear in a bestiary of equations and illustrations in the back of the book. This section includes not just the equation and the standard parameter settings, but also a short discussion of the equation's significance, and references to the literature. The value of this manual /reference book cannot be overemphasized.

The Low Down: If you're looking for computational power and a package that will grow with the student through more advanced follow-on courses, this probably is not the package for you. This is not a package for finding accurate answers to difficult problems, but if you want students with relatively little computer experience up and running on a package fast, this may be your best choice. You will need very little lecture time to explain the package.

If you want good software that can help your students learn about the geometry of solutions of ODEs and difference equations, software to introduce them to numerical methods, then **PHASER** may just be your ticket. ❑

Demonstrating the Stability of the Lagrange Points

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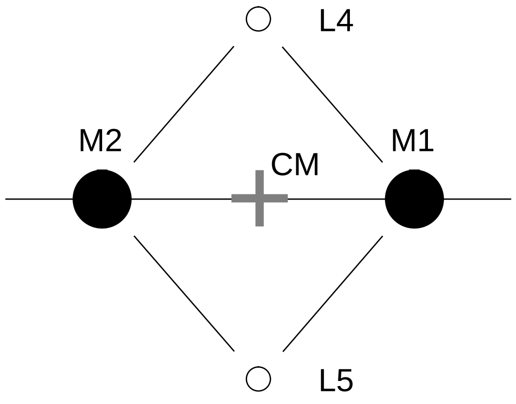
The goal of this lab is to make accessible to students of mechanics or differential equations the problem of determining the stability of motion of three bodies under mutual gravitational attraction. I have focused upon the restricted three-body problem, which is usually discussed in the later chapters of any text on analytical mechanics, such as Symon (**Mechanics**, Addison-Wesley, 1960).

Sun - Planet - Asteroid: The problem consists of describing the motion of a body of relatively small mass, M3, as it moves through the gravitational field of two bodies of significantly larger mass, M1 and M2, moving in circular orbits in a plane about their common center of mass (CM). An example of this would be the relationships between Jupiter, the sun, and the group of Trojan asteroids.

The problem is simplified by adopting a non-inertial frame of reference which rotates at the angular velocity of M1 relative to M2, so that the larger masses appear to be at rest. Here the small mass M3 appears to be under the influence of three forces: the gravitational attraction of M1 and M2, and a pseudoforce (‘centrifugal’ force) which is an artifact of the rotating frame.

Lagrange Points: Lagrange established that there are five points of equilibrium for M3. Three of these are collinear with M1 and M2 and are unstable, but the fourth and fifth (called L4 and L5) are stable under certain conditions. L4 and L5 form symmetric triangles with M1 and M2, as shown in Figure 1.

Figure 1

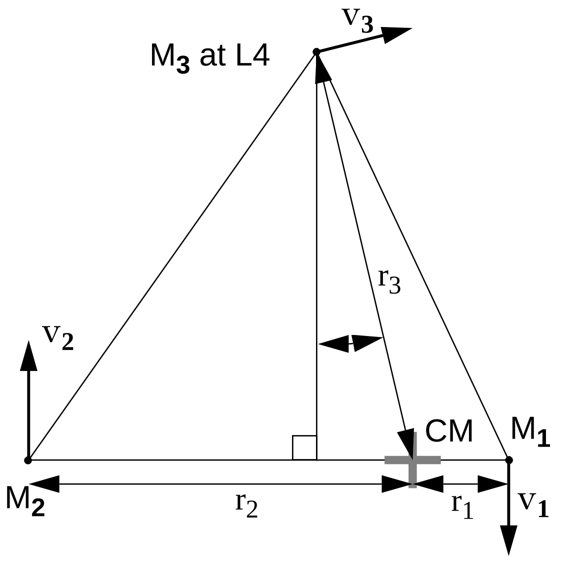


The small oscillations of M3 in the neighborhood of L4 or L5 may be approximated by solutions of three coupled linear second-order equations, if a non-inertial frame rotating with masses M1 and M2 is chosen. The derivation of these equations of motion may be found in Danby’s **Fundamentals of Celestial Mechanics** (MacMillan, 1962), along with a straightforward solution.

Plots of the effective potential suggest that stable oscillations of the exact (not linearized) equations are also possible (Danby, **Computing Applications to Differential Equations**, Reston, 1985). The oscillations may be investigated by generating numerical solutions of the nonlinear equations of motion using a program such as **PHASER**. (Ed. note: See review this issue!)

PLANETS, a program bundled with **MacMath**[1], offers a different approach to solving the problem. First, the orbits of the three bodies are observed in an inertial frame, so that all the objects are moving at once. Second, the equations of motion are encoded in the program but hidden from the user who simply selects different initial conditions and notes the effects.

Figure 2



Choosing initial conditions which illustrate the stability of L4 and L5 is facilitated by a knowledge of elementary mechanics. It is instructive to explore the problem with some guesswork before accepting the mathematics given below, if only to demonstrate the difficulty of finding stable three-body solutions. See Figure 2 for initial data.

Rewriting Kepler’s third law in terms of angular velocity ω and $a = r_1 + r_2$ we have

$$\omega^2 = \left(\frac{2\pi}{T}\right)^2 = \frac{G\,(M_1 + M_2)}{a^3}$$

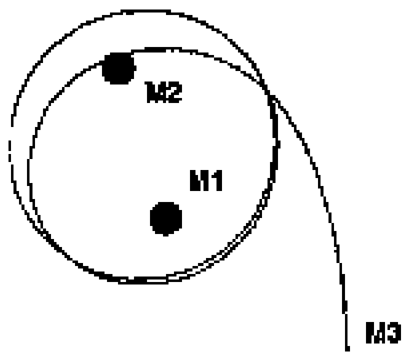
For circular motion, ω is constant; M1, M2, L4, and L5 rotate at a common uniform angular speed about the CM. Following Danby (op. cit.), we assume that $M_1 \geq M_2$, and introduce a dimensionless parameter.

$$m = \frac{M_2}{M_1 + M_2} \quad .$$

Using the definitions of the CM and the parameter m, the lengths of the position vectors of the three masses with respect to the CM are easy to find from the geometry.

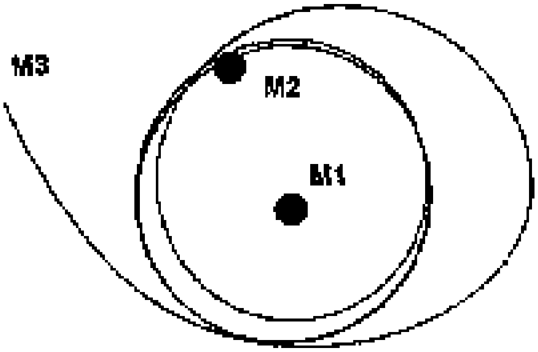
The values of the tangential velocities of M1, M2, and a small mass initially located at L4 are given by the simple relation $\mathrm{v} = r\omega$. If we start the large masses on the x-axis, v_1 and v_2 have only y-components. The x- and y-components of v_3 may be computed easily from the diagram in Figure 2.

Figure 3



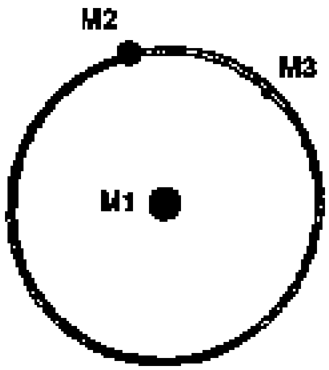
The results of some typical experiments testing the stability properties of L4 using **PLANETS** are given in Figures 3-7. Since L5 occupies a symmetric position with respect to L4, demonstrating the stability properties of one is adequate.The values entered for masses and distance are somewhat arbitrary. The total mass of the large objects, M1 and M2, was chosen to be 10^6 ; M3 (the small mass) is set to 1, and the semi-major axis is set to 1000. The units are chosen to make G = 1.

Figure 4



These values simplify the hand calculation of the initial positions and velocity components once the parameter m is chosen. It may be seen in Figures 3 and 4 that M3 does not remain at L4 if m is set to 0.5 or as low as 0.1, respectively. A value of m = 0.01 gives extremely stable behavior, however, as shown in Figure 5.

Figure 5



To simplify these diagrams, PLANETS was set to plot only the motion of M3. However, M1 and M2 continue to rotate as the solution proceeds, providing a real time movie! Danby (op. cit.) shows that the criterion for the stability of the Lagrange points is $m < 0.0385$.

Figure 6 shows the result of setting m = 0.04. M3 wobbles about L4 and then drifts slowly away.

A final PLANETS run (Figure 7) uses the setup in Figure 5, but this time the initial velocity values v_1 and v_2 are changed slightly so that the exact circular orbits of M1 and M2 become ellipses of low eccentricity. M3 oscillates about L4 in a leisurely fashion, with a substantial amplitude.

Figure 6

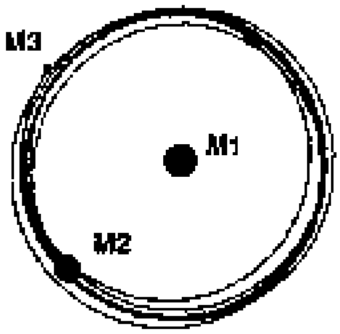
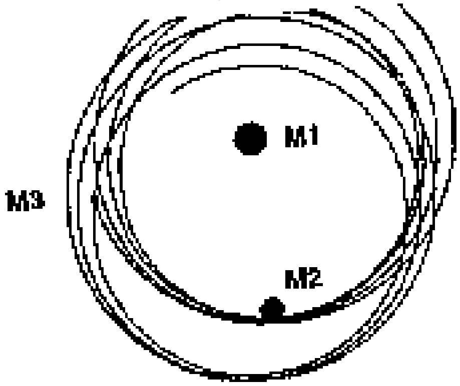


Figure 7



The demonstration described here is easy to carry out in the classroom, or could be used as a basis of student experiments with the restricted three body problem. The background may be easily adapted for a course in either ODEs or in classical mechanics. This lab is part of a project designed during a workshop on teaching ordinary differential equations at Cornell University.

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[1] **PLANETS** is included in the **MacMath** software distribution. For information, contact Beverly West at beverly@math.cornell.edu.ddd ❑

First Order Analysis of Communication Channels

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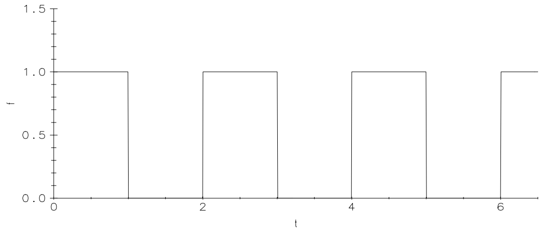
Introduction: The availability of computers certainly makes it easier for students to explore and investigate differential equations - especially nonlinear ones like the logistic equation. This is great, but the computer is also useful in helping us to reinforce what's going on with linear differential equations, even first order ones with constant coefficients.



Let us consider the bread and butter problem in electrical engineering of determining the maximum rate at which a train of data pulses (a square wave)can be sent down a communications channel modeled by a first order equivalent circuit of the form