

How to Balance a Stick When Walking in a Straight Line

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This project is motivated by the article, **How to Balance a Yardstick on an Apple**¹. In that article, the author evokes childhood memories of trying to balance such things as a stick, a broomstick, a baseball bat, etc. in the palm of his hand.

We seek to examine the behavior of a rod confined to move in a straight line over a level, smooth, frictionless surface [a mathematical surface]. In particular, we will ask the question: What horizontal moving force will keep a rod in a fixed angular position when the rod is deflected with respect to the vertical by a small angle? We begin with simpler problems, though, just to warm up a bit.

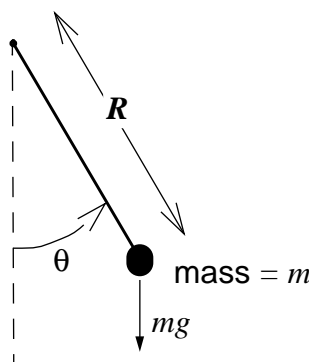
Newton's Law: The basic equation for rotation in a plane of a rigid body about a fixed axis is given by $T = I\ddot{\theta}$, where T is the net external torque (force times moment arm), θ is the angular acceleration, and I is the mass moment of inertia ($I = \int r^2 dm$, with dm an element of mass, and r its distance from the fixed axis). This relation is used in all of the models described below.

Simple Pendulum: A simple pendulum of mass m rotates in a vertical plane at a distance from a fixed axis. We assume the mass of the rod is small compared to the mass of the pendulum and ignore any air resistance (see Figure 1).

If we let $x = \theta$ and $y = \dot{\theta}$ then the corresponding first order system is given by the so-called phase-plane system, whose equations are given on the next page.

Figure 1

$$(1) \quad \begin{aligned} T &= I\ddot{\theta} \\ -mgR \sin \theta &= mR^2\ddot{\theta} \\ \ddot{\theta} + \frac{g}{R} \sin \theta &= 0 \end{aligned}$$



1. H. R. Bailey. **The College Mathematics Journal** 17(1986), 220-225.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{R} \sin x . \end{aligned} \tag{2}$$

When θ is small $\sin \theta \approx \theta$. In this case the nonlinear system (2) reduces to the simple linear harmonic case,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{R} x . \end{aligned} \tag{3}$$

Let $g = 32 \text{ ft/sec}^2$ and $R = 2 \text{ ft}$ in the following exercises.

Exercise 1. Find all equilibrium points in equation (2). Generate the phase plane and explain all equilibrium points. Plot several orbits. Are all orbits bounded? Explain these orbits in terms of specific motions of the pendulum.

Exercise 2. Find the equilibrium points in equation (3). Generate the phase plane, plotting a few orbits as above. Again, identify the nature of all equilibrium points. Superimpose the graphics generated in this exercise and the first one and comment on what you observe.

Inverted Pendulum: Suppose that the pendulum is inverted, that is, $\theta = 0$ corresponds to the pendulum standing straight up, with the mass directly above the pivot.

Exercise 3. Show that the differential equation for the inverted pendulum is

$$\ddot{\theta} - \frac{g}{R} \sin \theta = 0 . \tag{4}$$

(Hint: Replace θ with $\theta + \pi$ in the simple pendulum case.) What is the phase plane system for the inverted pendulum equation (4)? Find all equilibrium points and identify

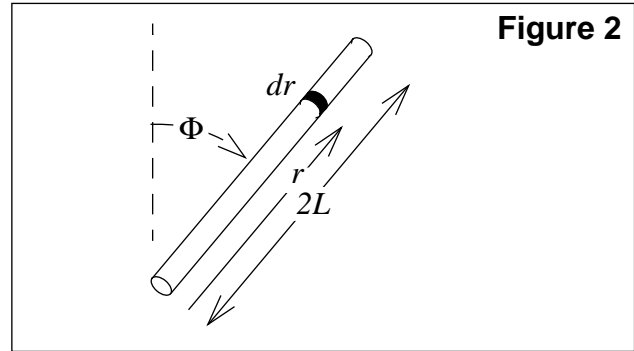


Figure 2

their nature. Plot some orbits, once again discussing the relationship of the orbits to pendulum motion.

Exercise 4. Suppose θ is small. Give the corresponding linear system for the inverted pendulum and find all equilibrium points. Generate the phase plane, plotting orbits and identifying the nature of all equilibrium points. How do these orbits correspond to the pendulum behavior?

Rotating Rod: Consider a rod of length $R = 2L$ rotating in a vertical plane about a fixed point at one end of the rod (see Figure 2). The equations of motion are:

$$\begin{aligned} T &= I \ddot{\phi} \\ \int_0^{2L} r \sin \phi \frac{m(r) g}{2L} dr &= \ddot{\phi} \int_0^{2L} r^2 \frac{m(r)}{2L} dr \\ mgL \sin \phi &= \frac{4mL^2}{3} \ddot{\phi} \\ \ddot{\phi} - \frac{3g}{4L} \sin \phi &= 0 \end{aligned}$$

where the third equation follows from the second under the assumption that the density is a constant.

In this case the corresponding system

when $x = \phi$ and $y = \dot{\phi}$ is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{3g}{4L} \sin x \end{aligned} \quad (5)$$

Once again, if we take ϕ small then the system of equations reduces to the linear system

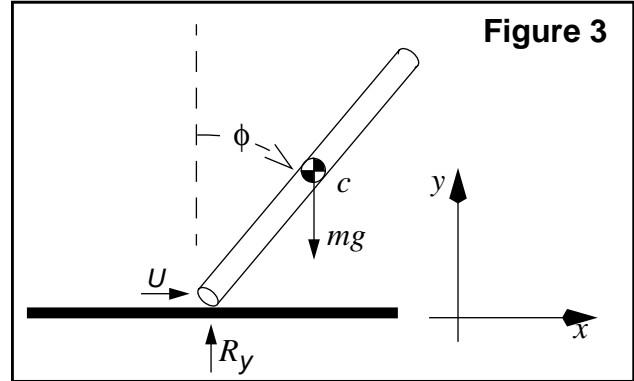
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{3g}{4L} x \end{aligned} \quad (6)$$

Exercise 5. Find all equilibrium points for the nonlinear case (5). Let $L = 4$, then generate the phase plane and classify all equilibrium points. Plot a few orbits and identify them with specific motions of the rod.

Exercise 6. Find all of the equilibrium points for the linear case (6). Once more, let $L = 4$ and generate the phase plane, classifying all equilibrium points by type. Plot a variety of different orbits and discuss their physical reality for the pendulum. Superimpose graphs from this exercise and the previous one, and then comment on what you see.

Balancing the Broom: Suppose a uniform rod of length $2L$ and mass m is free to move along a straight line in the vertical plane on a level, smooth, frictionless table. A horizontal force is applied to the base in the positive x direction. Let R_y denote the vertical reaction force of the table (see Figure 3) The equations of motion are:

$$T = I\ddot{\phi}$$



$$R_y L \sin \phi - UL \cos \phi = \frac{mL^2}{3} \ddot{\phi}$$

$$U = m\ddot{x}_c$$

$$R_y - mg = m\ddot{y}_c$$

where the torque equation has been applied at the center of gravity (x_c, y_c) ,

$$x_c = x + L \sin \phi$$

$$y_c = L \cos \phi$$

Using the equations for y_c , R_y , and the torque equation one can show that ϕ must satisfy

$$\ddot{\phi} = \frac{-\dot{\phi}^2 \sin \phi \cos \phi + \frac{g}{L} \sin \phi - \frac{U}{mL} \cos \phi}{\frac{1}{3} + \sin^2 \phi} \quad (7)$$

The corresponding system of first order equations is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{-y^2 \sin x \cos x + \frac{g}{L} \sin x - \frac{U}{mL} \cos x}{\frac{1}{3} + \sin^2 x} \end{aligned} \quad (8)$$

When ϕ is small, retaining only first order terms in it and its derivatives, equation (7) reduces to

$$\ddot{\phi} = \frac{3g}{L}\phi - \frac{3U}{Lm} \quad (9)$$

and system (8) reduces to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{3g}{L}x - \frac{3U}{Lm} \end{aligned} \quad (10)$$

We now seek to find a force function U that will stabilize the rod which is initially placed at some incline. What are appropriate functions to try?

Let $L = 2$ and weight be one pound.

Exercise 7. Let $U = a \tan x$, where a is a constant. Plot phase planes for $a = 0.5$, $a = 1.0$, and $a = 2$. Which graph suggests a solution to the standing rod problem? Try other choices of a and note the behavior.

Team Project. Try other functional forms for U . In particular, what happens if U is a linearized function of both ϕ and $\dot{\phi}$, that is if $U = ax + by$ in (10)? Check phase planes for various choices of constants a and b .

Hard Team Project. Try to find a force function U that will stabilize the upended stick using the full nonlinear system (8).

(Editor's Note: Ok, so this problem brings out the child in us! I tried to balance the custodian's push-broom, eventually succeeding (more or less...), but I couldn't do it by walking only in a straight line. I haven't a clue about what the balancing function was!) □

Can Terminal Velocity be Exceeded?

by "The Three Fallen Bodies"

James Fink

James H. Freeman

Charles Hampton

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(Editor's Note: Fink, Freeman, and Hampton have written a delightfully wry experiment that can be used early in a ODE course. We must caution you, however — your students might decide to perform a 'reality check' ...)

Purpose: To explore the problem of free-fall through the air. You will consider a variety of air resistance terms, graph the corresponding velocities, and compare results.

The Problem: Consider a freely falling body of mass m which falls under the action of gravity and is subject to air resistance. Our personal experience suggests that air resistance depends on the velocity of the body. If we let $f(v)$ denote the resistive force on the body when its velocity is $v(t)$ at time t , Newton's law of motion, $F = d(mv)/dt$, tells us that the initial value problem for the velocity is

$$m \frac{dv}{dt} = -mg + f(v), \quad v(0) = v_0$$

where g is the acceleration due to gravity and v_0 is the initial velocity. The positive direction is up; consequently, $v_0 = 0$ if the body is dropped from rest, $v_0 > 0$ if the body