When ϕ is small, retaining only first order terms in it and its derivatives, equation (7) reduces to

$$\ddot{\phi} = \frac{3g}{L}\phi - \frac{3U}{Lm} \qquad (9)$$

and system (8) reduces to

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{3g}{L}x - \frac{3U}{Lm} \end{aligned} \qquad (10)$$

We now seek to find a force function U that will stabilize the rod which is initially placed at some incline. What are appropriate functions to try?

Let $L = 2$ and weight be one pound.

Exercise 7. Let $U = a \tan x$, where a is a constant. Plot phase planes for a = 0.5, a = 1.0, and a = 2. Which graph suggests a solution to the standing rod problem? Try other choices of a and note the behavior.

Team Project. Try other functional forms for U. In particular, what happens if U is a linearized function of both ϕ and $\dot{\phi}$, that is if $U = ax + by$ in (10)? Check phase planes for various choices of constants a and b.

Hard Team Project. Try to find a force function U that will stabilize the upended stick using the full nonlinear system (8).

(Editor's Note: Ok, so this problem brings out the child in us! I tried to balance the custodian's push-broom, eventually succeeding (more or less...), but I couldn't do it by walking only in a straight line. I haven't a clue about what the balancing function was!) ❑

Can Terminal Velocity be Exceeded?

by "The Three Fallen Bodies"

James Fink

James H. Freeman

Charles Hampton

Mathematicians at Law

(Editor's Note: Fink, Freeman, and Hampton have written a delightfully wry experiment that can be used early in a ODE course. We must caution you, however — your students might decide to perform a 'reality check' …)

Purpose: To explore the problem of free-fall through the air. You will consider a variety of air resistance terms, graph the corresponding velocities, and compare results.

The Problem: Consider a freely falling body of mass m which falls under the action of gravity and is subject to air resistance. Our personal experience suggests that air resistance depends on the velocity of the body. If we let $f(v)$ denote the resistive force on the body when its velocity is $v(t)$ at time t, Newton's law of motion, $F = d(mv)/dt$, tells us that the initial value problem for the velocity is

$$m\frac{dv}{dt} = -mg + f(v), \qquad v(0) = v_0$$

where g is the acceleration due to gravity and v_0 is the initial velocity. The positive direction is up; consequently, $v_0 = 0$ if the body is dropped from rest, $v_0 > 0$ if the body is thrown upward, and $v_0 < 0$ it the body is thrown downward.

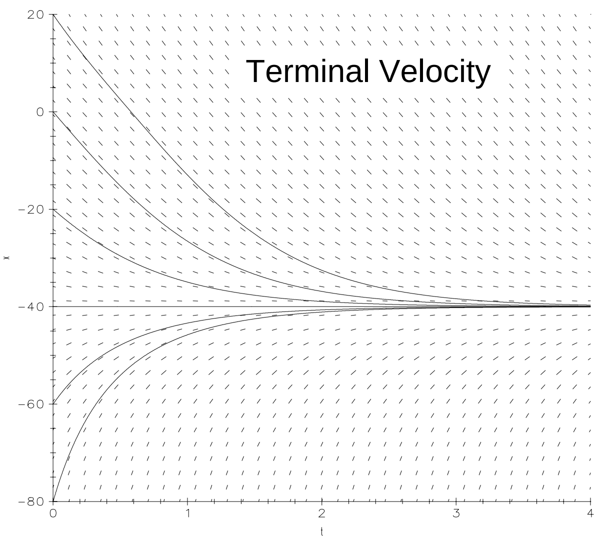
The nature of the air resistance function *f(v)* is very complex. Physical experiments indicate that air resistance is proportional to v for small v and to v^2 for large v. The size and shape of the body are factors as well, but since we will be working with only one body, these factors can be disregarded. This investigation will focus on the velocity of the falling body for various air resistance functions, starting with simple ones and progressing to more realistic – and complicated – functions.

Tasks: In the following, suppose the freely falling body is a 160 pound mathematician (m = 5 slugs). Since $g = 32 \text{ ft/sec}^2$ in the English system, the differential equation of motion becomes

$$\frac{dv}{dt} = -32 + \frac{1}{5} f(v) .$$

We begin by assuming that the air resistance function *f(v)* is proportional simply to v, say $f(v) = -v$ (the negative sign guarantees that the air always imparts a retarding force). Plot the velocity *v(t)* if the mathematician is dropped from rest ($v_0 = 0$). What happens as t gets large? The value $\lim_{t \to \infty} v(t)$, if it exists, is called the ***terminal velocity***. Find it for this system. Now plot the velocity for several negative and positive initial velocities. Does the terminal velocity change, or does it appear to be independent of the initial conditions?

Explain why the choice $f(v) = v$ in the first part does not give plausible results for any choice of initial velocities. In particular, explain what would happen if v_0 <160 ft/sec, v_0 =160 ft/sec, and v_0 >160ft/sec.



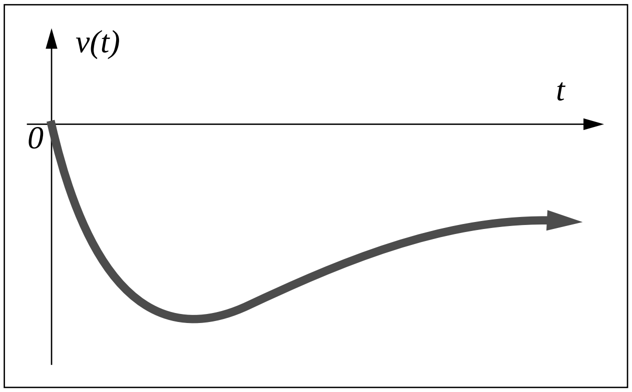
Repeat the previous exercise, but assume that the air resistance is proportional to v^2. Use the form $f(v) = -Cv|v|$, C a positive constant. The figure shows velocity curves if $f(v) = -0.1v|v|$.

Suppose you had taken $f(v) = -Cv^2$ or $f(v) = Cv^2$, C a positive constant. Why are these two forms of $f(v)$ unrealistic? Explain this.

Time to Play: In the beginning you looked at functions *f(v)* which yield reasonable models for air resistance. Now let's play with the function *f(v)* without any concern for physical reality - which seems appropriate since our falling body *IS* a mathematician!

Specifically, choose two non-polynomial functions for *f(v)* which are qualitatively different for each other. For each function, plot v against t for several initial conditions. Explain the qualitative behavior of the solutions for each *f(v)*. In particular, discuss whether for certain choices of initial condition your *f(v)* results in a terminal velocity, an unbounded velocity, or any strange behavior in the solution. Try to explain why *f(v)* produces the observed solution and what the resulting motion of the falling mathematician would look like.

Think Carefully: Do you think it is possible to choose an *f(v)* which results in a graph of v versus t like that shown here?



In other words, is it possible for the falling mathematician to "exceed" terminal velocity?

Functions of Time Too! Finally, suppose our mathematician is falling through the air with wild updrafts and downdrafts. In this case, air resistance becomes a function of both v and t. Pick your own function *f(v,t)* for air resistance, and see what strange behaviors you can generate. For example, can you find a function *f(v,t)* for which the mathematician falls for a while and then starts to rise again? This might happen if the mathematician gets caught in a strong updraft. Be sure to keep a record of the functions you try and the results you obtain.

Instructors' Notes: This laboratory exercise is intended for students early in a first ODE course. The exercise will require a software package that supports graphical output. The students may have trouble finding the appropriate bounds for the systems, so guidance is suggested. The kinds of student responses to the questions posed in this lab are very much dependent on whether the students have been introduced to the ideas of equilibrium and the existence and uniqueness theorem. In particular, the "Think Carefully" problem requires an understanding of these matters. ❑

Book Review:

V.I. Arnol'd 's *Ordinary Differential Equations*

Stephen Kennedy

(Editor's Note: Kennedy reviews the third edition of one of the classic texts in the field, written by V.I. Arnol'd, and published by Springer-Verlag, 1992. The translation was done by R. Cooke, and the book sells for $49.00)

Near the turn of the century, Poincaré, trying to understand the n-body problem, began the qualitative study of differential equations, i.e., studying the stability, topology, and asymptotics of solutions rather than searching for analytical expressions for them. The Russian school of dynamical systems, beginning with Poincaré contemporary Lyapunov and continuing through Andronov, Pontryagin, Kolmogorov and Arnol'd, developed a beautiful, mature qualitative theory rooted in mechanics with a strong geometric flavor.