Interactions and Reactions

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(Editor's Note: These three experiments are drawn from chemistry, ecology, and biology. They should encourage your students to see applications for differential equations outside their math books.)

Mutualism: A central theme in ecology is the study of interactions between pairs of species. These interactions may be such that species A benefits and species B is harmed (Predator-Prey), that both species are harmed by the presence of the other (Competition), or that both species benefit from the presence of the other (Mutualism). In this experiment, you will examine a model for two mutualists developed by May[1].

The model equations are:

$$dx/dt = rx\left[1 - \frac{x}{(K + ay)}\right]$$

$$dy/dt = sy\left[1 - \frac{y}{(L + bx)}\right] \quad .$$

Exploration: Explain why this model might represent an ecological system of two interacting species, each of which benefits from the presence of the other.

Find all steady-states (i.e., equilibrium points) of the model. Explain why we require $ab < 1$.

Obtain the linearized system at each equilibrium point. Determine the type of equilibrium and whether or not it is stable.

Choose some reasonable values for the parameters r,s,K,L,a, and b; now use your computer package to plot a phase-plane diagram of the original model, scaling the axis so that all equilibrium points are included. Interpret each orbit in terms of the populations.

Experiment with increasing (or decreasing) the values of r, K or a. In each case try to summarize how the equilibrium of each of the two species is influenced. Use phase-plane plots and explain the long-term population behavior for each parameter set you use.

Figure 1

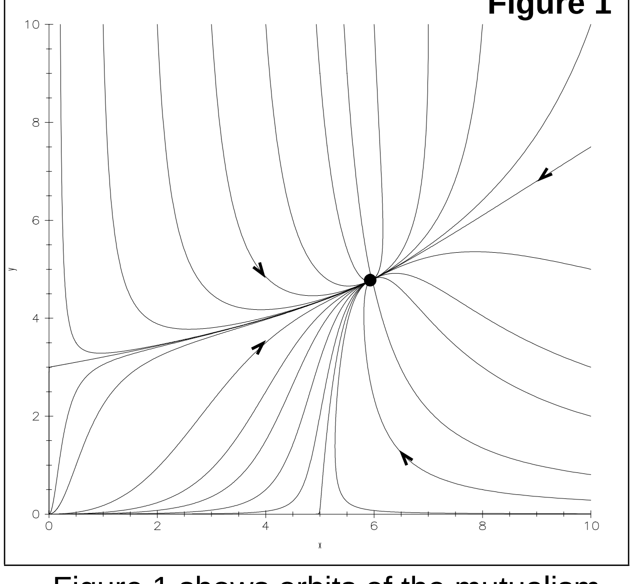


Figure 1 shows orbits of the mutualism system where the parameters are $r = 0.1$, $K = 5$, $a = 0.2$, $s = 0.2$, $L = 3$, and $b = 0.3$. The asymptotically stable node inside the population quadrant is the dominant feature.

Just Fishin': Now, suppose you are interested in modeling the interaction

1. R. M. May. *Models for Interacting Populations*. Chapter 4 of **Theoretical Ecology, Principles, and Applications**, R. M. May (ed.). Saunders Co., Philadelphia, 1976.

between fish and fishermen at a local resort lake. Let us assume that the fish population changes logistically and that fishing reduces the population at a rate jointly proportional to the population of both fish and fishermen. Suppose, also, that fishermen are drawn to the lake at a rate proportional to the fish population and leave at a rate proportional to the number of fishermen already present. Denoting the fish population by F and the fisherman population by H, we arrive at the following model for the fish-fisherman interaction:

$$\frac{dF}{dt} = rF\left(1-\frac{F}{K}\right) - aFH$$
$$\frac{dH}{dt} = bF - sH \quad .$$

Exploration: Find the equilibrium points for this model and characterize their stability.

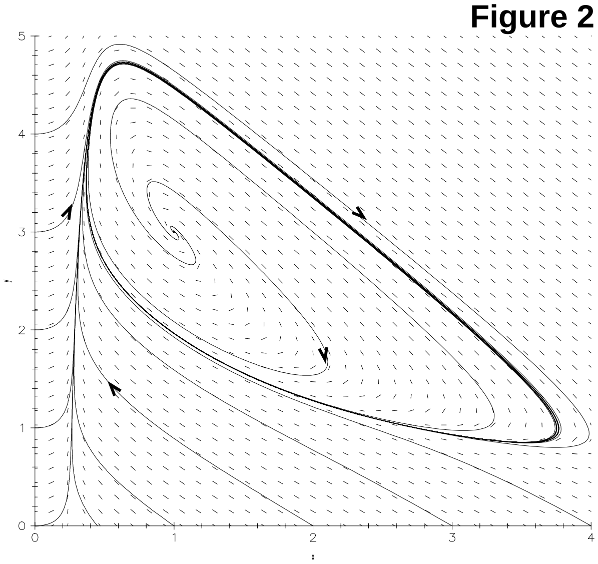
Construct a phase-plane diagram for this model using reasonable parameter values. Interpret each orbit in terms of what happens to the fish and the fishermen.

Suppose the fish are stocked at a constant rate S. Simulate the model for increasing values of S. What conclusions would you make about the effect of stocking the lake? Try to verify your conclusions analytically using phase-plane arguments.

Group Project: Explore the behavior of this model when the natural growth rate of the fish is not logistic. Some possibilities for a different natural growth rate are $rF\exp(-cF)$, $rF/(c+F)$, and $rF(1+cF-dF^2)$. These rate terms are, respectively, the Ricker, Beverton-Holt, and Allee models[2]. Does the new rate term change your conclusions about stocking the lake? Alternatively, consider the model when $S<0$; i.e., fish are removed at a constant rate.

Plot orbits inside the population quadrant for each case, and interpret the graphs in terms of the long term population behavior of both species.

Figure 2



The Brusselator: The potential for diffusive instability to generate Turing patterns is an important aspect of the analysis of particular chemical reaction systems. Such behavior cannot occur in the absence of diffusion, if the system exhibits oscillatory behavior. In this exploration we shall examine the Brusselator reaction first studied by Prigogine and Lefever[3]; this system exhibits stable oscillations over an appropriate parameter range (see Figure 2).

The Brusselator reaction is an idealized chemical system given by the following relations:

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2. L. Edelstein-Keshet. **Mathematical Models in Biology**. Random House, New York, 1988.
3. I. Prigogine and R. Lefever. *Symmetry Breaking Instabilities in Dissipative Systems, II.* **Journal of Chemical Physics** 48(1968), 1695-1700.

$$A \to X$$

$$2X + Y \to 3X$$

$$B + X \to Y + D$$

$$X \to E$$

When the concentrations of the species A, B, D, and E are assumed to be kept constant, a nondimensional version of the model is given by the following differential equations in the concentrations x and y:

$$\frac{dx}{dt} = 1 - (b+1)\,x + kx^2y$$

$$\frac{dy}{dt} = bx - kx^2y \quad .$$

Exploration: Find the equilibrium point of this model and determine conditions on b and k for the system equilibrium to be stable.

Examine the discriminant of the characteristic equation and determine bounds on b as a function of k, for which the roots are complex.

Using k as the independent variable and b as the dependent variable, graph (in the $k - b$ plane) the stability condition found above as well as the curves where the discriminant of the characteristic equation changes sign. Explain the relationship of these graphs.

Take a fixed value of $k \geq 1$ and explore the phase plane of this model for various values of b. Turn in phase-plane graphs and explanations (in terms of the long term concentrations of the two species) for the various results obtained. Figure 2 shows the attracting limit cycle of the concentrations if b = 3 and k =1. ❑

The Solow Model of Economic Growth

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(Editor’s Note: A mathematical model provides at best a rough approximation of reality, and this is particularly so with mathematical models of economic growth. Nevertheless, they do serve as useful guides for thinking about the arcane “laws” of economics. Solow’s models are among the best known, most used, and most criticized. Professors Johnson and Holden have split their intriguing approach to the simplest of Solow’s models into three modules. The first could be assigned as homework and the second as a computer laboratory experiment. The last module would make a great open-ended project. Caution: students are expected to have access to a computer algebra system such as Derive, Mathematica, or Maple to find solution formulas. Hand calculation will do the job as well, but the algebraic manipulations can be daunting.)

Preliminary Session: The production of goods and services by a country is assumed to depend on the capital available for investment and the number of workers in the labor force. If Q is the production output, K the capital, and L the labor force, then Q will be a function of K and L: $Q = f(K, L)$. Robert Solow developed a model for predicting Q that involves differential equations. In this module the student will examine the development of the differential