$$A \to X$$

$$2X + Y \to 3X$$

$$B + X \to Y + D$$

$$X \to E$$

When the concentrations of the species A, B, D, and E are assumed to be kept constant, a nondimensional version of the model is given by the following differential equations in the concentrations x and y:

$$\frac{dx}{dt} = 1 - (b+1)x + kx^2y$$

$$\frac{dy}{dt} = bx - kx^2y \quad .$$

Exploration: Find the equilibrium point of this model and determine conditions on b and k for the system equilibrium to be stable.

Examine the discriminant of the characteristic equation and determine bounds on b as a function of k, for which the roots are complex.

Using k as the independent variable and b as the dependent variable, graph (in the $k - b$ plane) the stability condition found above as well as the curves where the discriminant of the characteristic equation changes sign. Explain the relationship of these graphs.

Take a fixed value of $k \geq 1$ and explore the phase plane of this model for various values of b. Turn in phase-plane graphs and explanations (in terms of the long term concentrations of the two species) for the various results obtained. Figure 2 shows the attracting limit cycle of the concentrations if b = 3 and k =1. ❑

The Solow Model of Economic Growth

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(Editor's Note: A mathematical model provides at best a rough approximation of reality, and this is particularly so with mathematical models of economic growth. Nevertheless, they do serve as useful guides for thinking about the arcane "laws" of economics. Solow's models are among the best known, most used, and most criticized. Professors Johnson and Holden have split their intriguing approach to the simplest of Solow's models into three modules. The first could be assigned as homework and the second as a computer laboratory experiment. The last module would make a great open-ended project. Caution: students are expected to have access to a computer algebra system such as Derive, Mathematica, or Maple to find solution formulas. Hand calculation will do the job as well, but the algebraic manipulations can be daunting.)

Preliminary Session: The production of goods and services by a country is assumed to depend on the capital available for investment and the number of workers in the labor force. If Q is the production output, K the capital, and L the labor force, then Q will be a function of K and L: $Q = f(K, L)$. Robert Solow developed a model for predicting Q that involves differential equations. In this module the student will examine the development of the differential equation model from the assumptions phase through the solution, analysis, and interpretation phases.

Assumptions: Any mathematical model is based on assumptions that the modeler makes to balance the complexity of the phenomenon with the tractability of the model. The assumptions of Solow's model are based on observations of actual economic conditions.

Assumption 1: The production function is dependent on the amount of capital per worker and is proportional to the number of workers: $Q = L\ g(K/L)$.

Assumption 2: The labor force changes exponentially in time with growth rate α per unit of labor and initial value L_0.

Assumption 3: The growth rate of capital is proportional to production.

Task 1. Explain in your own words why you believe that the assumptions are reasonable. Hint: Some key economic words for Assumption 1 are "return to scale". For #2 think of growth of a population; for #3 think of investing a certain percentage of income.

Task 2. Use Assumption 2 to write L as a function of time.

Task 3. Assumption 3 can be modeled by the differential equation, $dK/dt = sQ$. Use the function from task 2 to obtain the differential equation,

$$\frac{dK}{dt} = sL_0 \exp(\alpha t)\, g\left(\frac{K}{L}\right) .$$

Task 4. Let $y = K/L$. Differentiate $K = yL_0 \exp(\alpha t)$ by the product rule to obtain

$$\begin{aligned}\frac{dK}{dt} &= yL_0 \alpha \exp(\alpha t) + \frac{dy}{dt} L_0 \exp(\alpha t) \\ &= \left(y\alpha + \frac{dy}{dt}\right) L_0 \exp(\alpha t) .\end{aligned}$$

Use this expression for dK/dt and that found in task 3 to obtain $dy/dt = s g(y) - \alpha y$.

Task 5. The differential equation for y in task 4 can be solved for suitable functions g by separation of variables and integration. Do this by hand or with a Computer Algebra System for $g(y) = y^2$. Remember that you have to find y as a function of t.

Computer Laboratory Session: For convenience the differential equation for y is repeated: $dy/dt = sg(y) - \alpha y$. This equation can be solved for t by integration:

$$t + C = \int \frac{1}{s\, g(y) - \alpha y}\, dy .$$

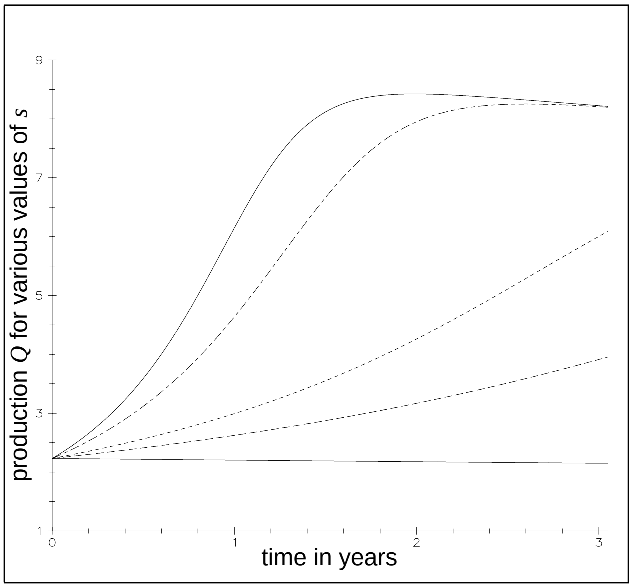
As you have discovered in task 5, solving for y as a function of t is not easy. In this session you will find y for functions g that make the differential equation for y a Bernoulli equation, which can then be converted to a first order linear differential equation by a change of variable. Use a Computer Algebra System if one is available.

Task 6. Solve the differential equation for y using initial condition $y_0 = 0.5$, $\alpha = 0.03$, and $s = 0.2$. For $g(y)$ use **(a)** $\sqrt{y}$, **(b)** $\sqrt{y^3}$, **(c)** y^3, **(d)** y. Be sure to verify that you have a Bernoulli equation.

Task 7. Explain what happens in case **6(d)**. Warning: Using the Bernoulli command in **Derive** may cause problems!

Task 8. Graph your solutions either by using a graphing program or by using a differential equations solver/graphics package.

Task 9. You desire sustained growth of Q, the quantity of goods and services. Since $Q = Lg(y)$, you want g to increase with y (for $y > 0$) and y to increase with t. Which of the forms of g given in task **6** satisfy these conditions?



Project: What advice do you give President Clinton today to improve his chances for re-election in 1996? What would you have told President Bush in 1989 so that the preceding question would be irrelevant?

As chief economic adviser to the President of United States you are concerned about the welfare of the nation and the re-election of the President three years from now. You believe that the best strategy to achieve both is to insure that the production of goods and services (i.e., the function Q) is increasing. This means that the parameters must be adjusted so that $dQ/dt > 0$. You have no control over α, which is currently -0.03 (not the 0.03 used earlier). However, you can control the reinvestment rate s by certain tax incentives. Your job is to find a value for s that will lead to $dQ/dt > 0$ when $t \approx 2.5$ (i.e., shortly before the next presidential election). You will have to convince the President that your recommendations are reasonable, so prepare graphs of Q(t), $0 \le t \le 3$, for various assumed forms for $g(y)$ and various values of s, α, and L_0.

Hints: Use the chain and product rules to obtain the differential equation,

$$\frac{dQ}{dt} = s\, g(y)\, g'(y) + \alpha(L_0 \exp(\alpha t)\, g(y) - y\, g'(y)).$$

Use $g(y) = y^n$ and simplify the differential equation. Set $n = 1/2$ for a specific model and L_0 = 3. Find s so that Q is increasing if $t = 2.5$ years.Examine the behavior of dQ/dt for differing values of L, α, and s. What happens as t tends to infinity? Repeat with other values of n and L_0. Try other forms of $g(y)$ for which it is not possible to solve for y in terms of t, but for which you can get graphs by using differential equations solvers and graphics. For example, try

$$g = \frac{a}{1 + b\exp(-cy)}$$

which has a rising S-shaped graph. The figure accompanying this article corresponds to this g with a=3, b=5, c=1. The graphs show how Q changes as s is changed from 0.01, to 0.3, 0.5, 1.1, 1.5 (reading from bottom to top). Argue for or against each of these graphs, given your aims as the President's economic adviser. Incidentally, the graphs are obtained by using a computer solver to solve the differential equation for y and then using a graphics package to plot $Q = L\,g(y)$. ❑