Samurai Sword

Background. Using a trial-and-error approach over many centuries, Samurai warriors in Japan gradually developed a highly efficient cutting instrument, arguably the most effective sword the world has ever seen. The blade of a Samurai sword has the characteristically curved shape that appears in the photo below:

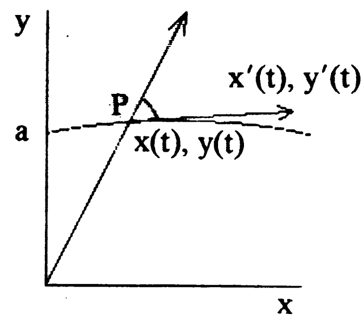
In modern times it was discovered that the shape of a Samurai sword is governed by one basic principle:

> The blade of the sword has a constant **angle of attack** as seen from a point in the plane of the sword.

We are indebted to Sam Saunders of Washington State University for bringing this modeling problem to our attention. Sam is an avid collector of Samurai swords.

Building the Model

Suppose that x(t), y(t) parametrically describe the sword's edge in the xy-plane and that x(0) = 0, y(0) = a. Choosing the origin as a viewpoint we have the diagram:



For any point P on the sword, ϕ is the angle of attack. Let's assume that $x'(0) > 0$, $y'(0) > 0$, and so $0 < \phi < \pi/2$.

Method 1. At every point P on the sword the vectors (x(t), y(t)) and (x'(t), y'(t)) must have the same angle of attack ϕ. Thus,

$$xx' + yy' = \cos\phi \sqrt{x^2+y^2}\sqrt{x'^2+y'^2}$$

Dividing by x' and recalling that $dy/dx = y'/x'$,

$$x + y\frac{dy}{dx} = \cos\phi \sqrt{x^2+y^2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Solving for dy/dx we have

$$\frac{dy}{dx} = \frac{-xy+\cos\phi\sin\phi(x^2+y^2)}{\sin^2\phi\, y^2 - \cos^2\phi\, x^2} = \frac{\cos\phi\, y - \sin\phi\, x}{\sin\phi\, y + \cos\phi\, x} \qquad (1)$$

Method 2. Rotating the unit vector in the (x, y) direction into the unit vector in the (x',y') direction, we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \frac{\sqrt{x'^2+y'^2}}{\sqrt{x^2+y^2}}$$

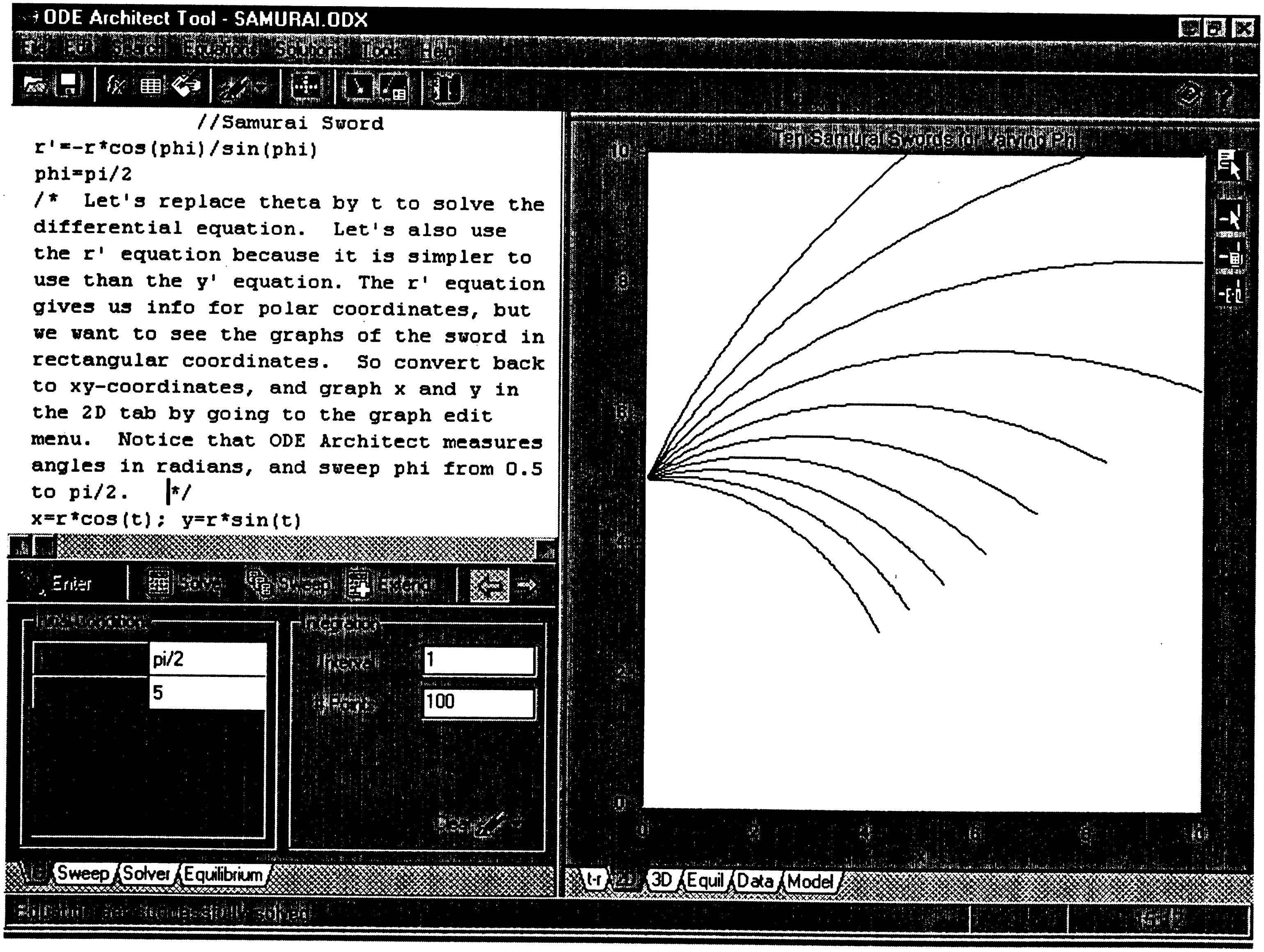
Dividing the y' equation by the x' equation yields ODE (1). Converting to polar coordinates,

$$\frac{dr}{d\theta} = -\frac{\cos\phi}{\sin\phi} r, \qquad r\left(\frac{\pi}{2}\right) = a$$

whose solution is

$$r = a e^{-(\cos\phi/\sin\phi)(\theta - \frac{\pi}{2})}$$

which is an arc of a spiral.


ODE Architect Tool - SAMURAI.ODX
//Samurai Sword
r'=-r*cos(phi)/sin(phi)
phi=pi/2
/* Let's replace theta by t to solve the differential equation. Let's also use the r' equation because it is simpler to use than the y' equation. The r' equation gives us info for polar coordinates, but we want to see the graphs of the sword in rectangular coordinates. So convert back to xy-coordinates, and graph x and y in the 2D tab by going to the graph edit menu. Notice that ODE Architect measures angles in radians, and sweep phi from 0.5 to pi/2. */
x=r*cos(t); y=r*sin(t)
Enter
pi/2
5
1
100
Sweep
Solver
Equilibrium
Ten Samurai Swords for Varying Phi
t-r
3D
Equil
Data
Model